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# Strong-coupling expansion of cusp anomaly from quantum superstring

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Abstract: We consider the world surface in $AdS_5$ that ends on two intersecting null lines at the boundary. The corresponding superstring partition function describes the expectation value of the Wilson line with a null cusp in dual large $N$ maximally supersymmetric gauge theory and thus determines the cusp anomaly function $f(\lambda)$ of the gauge coupling $\lambda$ or the string tension $\frac{\sqrt{\lambda}}{2\pi}$. The first two coefficients in its strong-coupling or string inverse tension expansion were determined in hep-th/0210115 ($\mathrm{a}_0 = 1$) and in arXiv:0707.4254 ($\mathrm{a}_1 = -3\ln 2$). Here we find that the 2-loop coefficient is $\mathrm{a}_2 = -\mathrm{K}$ where K is the Catalan's constant. This is in agreement (expected on the general grounds) with the previous results for $f(\lambda)$ as the coefficient of $\ln S$ term in the energy of the closed spinning string in $AdS_5$. The string theory value for $\mathrm{a}_2$ is in agreement with the numerical result in hep-th/0611135 and the recent analytic result in arXiv:0708.3933 for the coefficients in strong-coupling Ssolution of the BES equation. We explicitly verify the cancellation of all 2-loop logarithmic divergences thus demonstrating the quantum consistency of the $AdS_5 \times S^5$ superstring action at this order. We also discuss the structure of the three and higher string loop corrections to the cusp anomaly function giving a 2d QFT diagrammatic interpretation to the result of arXiv:0708.3933 for the solution of the BES equation following from the Bethe ansatz prescription for the spectrum of the theory.

Keywords: Field Theories in Lower Dimensions, AdS-CFT Correspondence.

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## Contents



# 1. Introduction

Anomalous dimension of minimal twist large spin single trace operator or anomalous dimension of a Wilson line with a null cusp [1] was a subject of much attention in the context of the AdS/CFT duality for several years starting with the seminal work of [2] (see also [3–5]). In the planar limit this dimension is a function $f(\lambda)$ of the 't Hooft coupling $\lambda$ or of the $AdS_5 \times S^5$ string tension $\frac{\sqrt{\lambda}}{2\pi}$. Finding this function exactly would be an important progress. A series of recent developments based on the apparent integrability of the theory culminated in a suggestion [6] of an integral equation that, in principle, determines $f(\lambda)$ for any value of $\lambda$.

To check the consistency of this equation and thus of the underlying asymptotic Bethe ansatz it is important compare its prediction with that of the quantum superstring theory in $AdS_5 \times S^5$ . The perturbative string theory or the strong-coupling expansion of $f(\lambda)$ can be written as

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} \left[ \mathrm{a}_0 + \frac{\mathrm{a}_1}{\sqrt{\lambda}} + \frac{\mathrm{a}_2}{(\sqrt{\lambda})^2} + \frac{\mathrm{a}_3}{(\sqrt{\lambda})^3} + \dots \right], \tag{1.1}$$

where the tree-level [2] and the 1-loop [3] superstring predictions are

$$\mathrm{a}_0 = 1\,, \qquad\qquad \mathrm{a}_1 = -3\ln 2\ . \tag{1.2}$$

The computation of the 2-loop superstring coefficient was initiated in [7][1] where it was found to be expressed in terms of the Catalan's constant $\mathrm{K} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159$.

[1]Note that in the notation of [7] $a_k = \frac{1}{\pi}\mathrm{a}_k$.



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The expansion of the BES [6] equation at strong coupling turned out to be a non-trivial problem [8 − 12].[2] The results for the three leading $\mathrm{a}_n$ coefficients (1.2) were first found only numerically [8] ($\mathrm{a}_0$ was later computed exactly [10]).[3] The numerical result for the third coefficient found in [8] was $\mathrm{a}_3 \approx -0.9158 \pm 0.0039$.[4]

Very recently the analytic results for the coefficients in the strong coupling expansion of the solution of the BES equation for the cusp anomaly function (1.1) was found in a remarkable paper of [18], with the first few leading coefficients given by[5]

$$\mathrm{a}_2 = -\mathrm{K}\,, \tag{1.3}$$

$$\mathrm{a}_3 = -\frac{1}{32}\big[27\zeta(3) + 96\mathrm{K}\ln 2\big]\ , \tag{1.4}$$

$$\mathrm{a}_4 = -\frac{1}{16}\big[84\beta(4) + 81\zeta(3)\ln 2 + 32\mathrm{K}^2 + 144\mathrm{K}(\ln 2)^2\big], \tag{1.5}$$

$$\mathrm{a}_5 = -\frac{9}{2048}\big[4785\zeta(5) + 10572\beta(4)\ln 2 + 4416\zeta(3)\mathrm{K} + 5184\zeta(3)(\ln 2)^2 + 4096\mathrm{K}^2\ln 2\big] \tag{1.6}$$

where

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}\,, \qquad \beta(k) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^k}\,, \qquad \beta(2) = \mathrm{K}\ . \tag{1.7}$$

The expression for $\mathrm{a}_2$ (1.3) thus agrees with the numerical value found in [8] and matches precisely (the corrected version of) the result of the 2-loop superstring computation in [7].

Our aim here is to confirm the Catalan constant value of $\mathrm{a}_2$ in (1.3) by an independent 2-loop superstring computation. The agreement of the results for $\mathrm{a}_2$ obtained in [8] and [18] from the BES equation with our superstring expression provides an important test of the BES equation and thus of the underlying asymptotic Bethe ansatz. The significance of the result of the present paper is that it provides a highly non-trivial confirmation of the proposal for the all-order strong-coupling phase [17] and its weak-coupling continuation in [6]. Indeed, while the expressions for the tree-level [19] and the 1-loop [20, 21] terms in the strong-coupling expansion for the phase where essentially put into the Bethe ansatz expression from the known string theory results, the higher order terms in the phase where conjectured in [17] using the crossing symmetry condition [22] (which so far was not directly derived from string theory). The present computation demonstrates that the 2-loop term in the phase suggested in [17] is indeed in agreement with string theory.

The computation described below also resolves a technical problem related to UV regularization present in the original approach of [7]. The manifest cancellation of the logarithmic UV divergences that we find here provides a direct demonstration of the quantum

[2]See also [13] for a potentially important alternative approach based on Baxter equation. A strong-coupling interpolation of the sum of few leading perturbative gauge-theory coefficients which appears to be in good agreement with the string results (1.2) was discussed in [14, 15, 9].

[3]$\mathrm{a}_1$ was also computed [16] from the "string" version of the Bethe ansatz, i.e. with the magnon scattering phase taken in the strong-coupling expanded form [17].

[4]The proximity of the absolute value of this number to the value of the Catalan's constant was noticed by the authors of [7] but the final result for the coefficient $\mathrm{a}_3$ in the original version of [7] was incorrect due to several errors which were finally corrected in the revised version ([7],v4).

[5]The relation of the notation used in [18] to ours is: $\Gamma_{\rm cusp}(g) = \frac{1}{2}f(\lambda)$, $c_k = -\frac{1}{(4\pi)^k}\mathrm{a}_k$, $g = \frac{\sqrt{\lambda}}{4\pi}$. We do not shift the argument of cusp anomaly function $\Gamma_{\rm cusp}(g)$ by $c_1$ as was done in [18].



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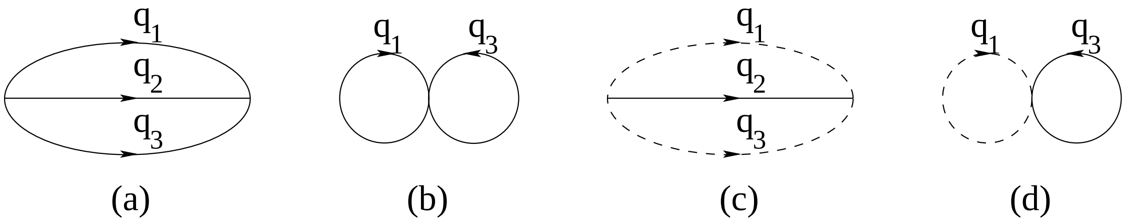


**Figure 1:** Two-loop diagrams (bosonic propagators are denoted by solid lines and fermionic ones are denoted by dashed lines).

consistency of the $AdS_5 \times S^5$ Green-Schwarz (GS) action of [23]. This (together with the earlier 1-loop results [3, 24]) removes any doubt that this action can be used as a basis for non-trivial strong-coupling computations in the AdS/CFT.

Another new result is the suggestion of a 2d Feynmann diagram (i.e. quantum superstring) interpretation to the higher-order coefficients (1.4)–(1.6), etc. found in [18]. In our computation $f(\lambda)$ appears in the quantum 2d effective action of the $AdS_5 \times S^5$ superstring sigma model expanded near a particular "homogeneous" string background in $AdS_5$

$$\Gamma = -\ln Z = \frac{1}{2} f(\lambda) V_2 \ . \tag{1.8}$$

$\Gamma$ is proportional to the (large) volume factor $V_2$.[6] This 2d QFT interpretation of $f(\lambda)$ implies that different parts of the transcendental coefficients $\mathrm{a}_L$ appearing in (1.1), (1.3)–(1.6) can be associated with the contributions of different $L$-loop Feynmann diagrams in the superstring sigma model.

In the 2-loop case both the bosonic and the fermionic "sunset" diagrams (figures 1a and 1c) happen to contribute terms proportional to K (see [7] and below). Extending our superstring computation to the 3-loop order appears to be relatively straightforward. A qualitative analysis shows that $\zeta(3)$ term in $a_3$ in (1.4) should originate from diagrams in figures 2b, 2e, 2g and 2h, while the $\mathrm{K} \ln 2$ term should come from diagrams in figures 2c and 2f. In general, it is natural to conjecture that the "maximally irreducible" terms $\zeta(2m+1)$ in the coefficients $\mathrm{a}_{2m+1}$ and $\beta(2m)$ in the coefficients $\mathrm{a}_{2m}$ [18] should originate, respectively, from the "maximally irreducible" odd-loop $L = 2m+1$ and even-loop $L = 2m$ superstring Feynman diagrams.[7]

This string world-sheet, i.e. 2d QFT interpretation of the function $f(\lambda)$ may help to clarify the meaning of the Borel non-summability of the strong-coupling expansion for $f(\lambda)$ as found from the BES equation in [18]. As was observed in [18], all coefficients $\mathrm{a}_k$ in (1.1) except the first one are *negative* and their values grow factorially (cf. (1.4)–(1.6)). It appears that in contrast to sign-alternating Borel-summable series usually found in QM or QFT problems with perturbatively stable vacuum here we are dealing with an expansion

[6] For a homogeneous backgrounds such as those considered in [7] and here there is no distinction between the 1-PI effective action and the logarithm of the partition function $Z$: connected but not 1-PI irreducible 2d Feynman graphs vanish.

[7] This should apply starting with 2-loop order. Using this logic at the 1-loop order one would get $\mathrm{a}_1 \sim \zeta(1)$ but this is logarithmically divergent; in fact, the 1-loop divergences cancel between bosons and fermions and the finite remainder happens to be proportional to $\ln 2$ [3, 25]. The 1-loop tadpoles adjoined to lower-loop topologies should perhaps be interpreted in this way.

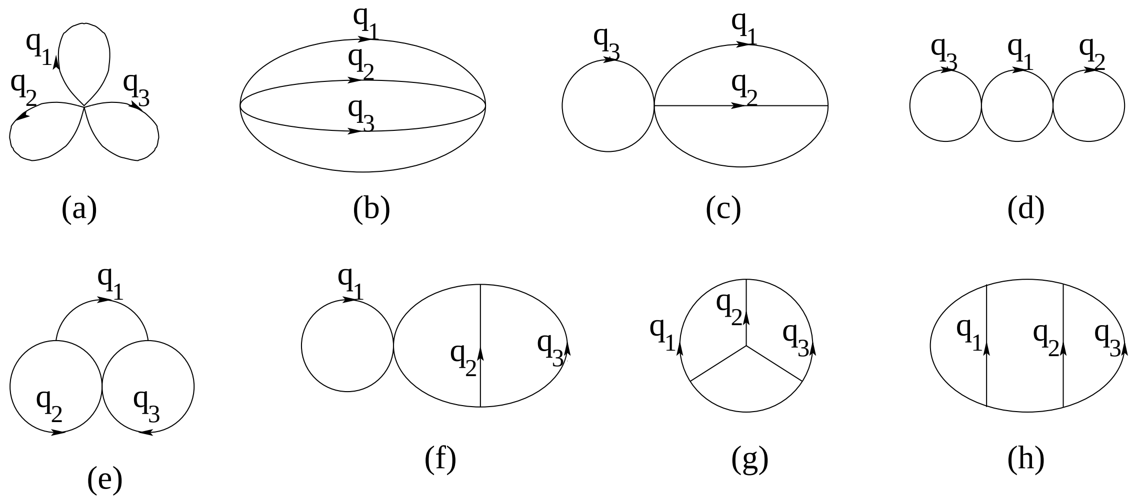


**Figure 2:** Topologies bosonic three-loop diagrams (diagrams with fermionic lines have similar topology).

near an unstable point. This is puzzling since the rotating folded string solution or the null cusp solution of [4] we consider below (which are closely related [25, 26]) are perturbatively stable.[8] One may contemplate the presence of some non-perturbative instability. We shall further comment on this in the concluding section 4.

We shall start in section 2 with setting up the computation of the the cusp anomaly function using the open-string (Wilson line [27, 28]) approach which is based on expansion near a Wilson line surface with a null cusp [4, 25]. As was explained in [25, 29, 26] it is equivalent to the closed-string approach used in [2, 3, 30, 7]. We shall use the $AdS_5 \times S^5$ GS superstring action in a special $\kappa$-symmetry gauge which becomes *quadratic* in fermions [31] after the T-duality along the 4 $AdS_5$ boundary directions in the Poincare coordinates.[9] This action was already used in [25] for the computation of the 1-loop coefficient $\mathrm{a}_1$ in (1.2). Here we shall utilize its simple structure (in particular, the absence of the quartic fermionic terms) to perform the computation of the 2-loop coefficient $\mathrm{a}_2$.

In section 3 we shall turn to computation of quantum corrections to string partition function expanded near the “null cusp” string background. We shall discuss the issue of UV regularization, pointing out that the structure of the superstring action involving the $\epsilon^{ab}$ tensor in the fermionic term prohibits the use of a direct version of the 2d dimensional regularization. Its use is not actually necessary since we find that all the logarithmic 2-loop

[8] In the conformal gauge we will be using here there is formally a ghost fluctuation mode corresponding to the time direction in $AdS_5$ but like in the flat Minkowski space case or in the $AdS_3$ WZW model the underlying string theory should be unitary: the Virasoro condition selects only physical on-shell modes. In our conformal-gauge partition function computation we are expanding near a consistent on-shell string background so the unphysical modes (a massless time-like (ghost) fluctuation mode and another massless longitudinal mode) should decouple and they actually do (their trivial 1-loop contribution cancels against that of the conformal gauge ghosts).

[9] This action was found in [31] by starting with the action of [23] written in a special $\kappa$-symmetry gauge discussed in [32]. An equivalent action which also becomes quadratic in fermions after the T-duality was found in a similar $\kappa$-symmetry gauge (“S-gauge”) in appendix C of [33].

divergences cancel out separately in the sums of the bosonic and fermionic graphs computed directly in $d = 2$. The remaining power divergences can then be eliminated using a kind of analytic regularization which essentially amounts to setting $\delta^{(2)}(0) = 0$.[10] This should be considered as a regularization prescription that defines the quantum $AdS_5 \times S^5$ superstring theory in a way consistent with its classical symmetries, i.e. as a conformal quantum 2d field theory.

As summarised in section 3.2, the resulting finite contributions to the 2-loop coefficient in (1.1) coming from the bosonic and from the fermionic 2-loop graphs in figure 1 happen to be the same as found in the closed-string picture computation in [7]

$$\mathrm{a}_2 = \mathrm{a}_{2B} + \mathrm{a}_{2F} = \mathrm{K} - 2\mathrm{K} = -\mathrm{K}\,, \tag{1.9}$$

so that the total result matches the value in (1.3). Higher-loop generalizations are discussed in section 3.3.

Section 4 contains some remarks on the problem with summability of the series in (1.1) and also on a possible generalization of the present 2-loop computation to the case of non-zero angular momentum $J$ in $S^5$.

Some technical details related to the structure of the fluctuation Lagrangian from section 3 are given in appendix.

## 2. Superstring action, classical string background and fluctuations

Our starting point will be the path integral with the euclidean version of the $\kappa$-symmetry gauge fixed and T-dualized $AdS_5 \times S^5$ action found in [31]. This action is remarkably simple being quadratic in fermions ($m = 0, 1, 2, 3;\ s = 4, \ldots, 9,\ \ z^2 = z^s z^s,\ a, b = 0, 1$)[11]

$$S_E = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[ \frac{1}{z^2}(\partial^a x^m \partial_a x_m + \partial^a z^s \partial_a z^s) + 4\epsilon^{ab}\bar{\theta}(\partial_a x^m \Gamma_m + \partial_a z^s \Gamma_s)\partial_b \theta \right]\,. \tag{2.1}$$

Here $\theta$ is a Majorana-Weyl 10d spinor and $\Gamma_A$ are standard "flat" 10d Dirac matrices. For our present purpose the use of this T-dual action is a technical trick that allows us to reduce the number of fermionic 2-loop diagrams we should compute.[12]



[10] In principle, one should be able to show the cancellation of all power-like divergent terms directly, by carefully including the contributions of all local factors (measure, $\kappa$-symmetry ghosts, Jacobians due to change of fluctuation bases, etc.). Bosonic power-like divergences are indeed cancelled by the invariant measure contribution [7]. The same should apply to the fermionic sector: as was discussed in appendices C and D.1 in [7], the cancellation of the 2-loop power-like divergences is required in order for the superstring partition function to be equal to 1 in supersymmetric cases such as the flat space GS action expanded near a long fundamental string background and the $AdS_5 \times S^5$ GS action expanded near a BMN geodesic.

[11] We shall mostly follow the notation of [25]. We choose the conformal gauge and ignore the dilaton coupling originating from the 2d duality transformation. We also use Euclidean signature on the world sheet, i.e. $\sigma^a = (\sigma^0, \sigma^1)$ ($\sigma^0 = -i\tau$) as appropriate for the null cusp solution of [4]; thus there is no $i$ in front of the fermionic term.

[12] This T-duality is a quantum symmetry when both world sheet directions are non-compact (as is the case in our present discussion). The T-duality maps the bosonic $AdS_5 \times S^5$ part of the action into an equivalent $AdS_5 \times S^5$ bosonic sigma model ($z \to z^{-1}$ is a symmetry transformation). Thus the bosonic

It will be useful to split the 6 coordinates $z^s \equiv z\hat{z}^s$, $\hat{z}^2 = 1$ orthogonal to the directions $x^m$ along the boundary of $AdS_5$ as ($s' = 4, \ldots, 8$)

$$z^{s'} \equiv z\hat{z}^{s'} = z\frac{y^{s'}}{1+\frac{1}{4}y^2}\,, \qquad z^9 \equiv z\hat{z}^9 = z\frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}\,, \qquad \frac{dz^s dz^s}{z^2} = \frac{dz^2}{z^2} + \frac{dy^{s'}dy^{s'}}{\left(1+\frac{1}{4}y^2\right)^2} \tag{2.2}$$

where $y^{s'}$ parametrize $S^5$.

### 2.1 "Null cusp" solution

The conformal-gauge form of the solution for the open string world sheet ending on two light-like lines forming a cusp at the boundary is [4, 25]

$$\bar{z} = \sqrt{2}\, e^{-\alpha\sigma_0 - \beta\sigma_1}\,, \tag{2.3}$$

$$\bar{x}^0 = e^{-\alpha\sigma_0-\beta\sigma_1}\cosh(\beta\sigma_0 - \alpha\sigma_1)\,, \qquad \bar{x}^1 = e^{-\alpha\sigma_0-\beta\sigma_1}\sinh(\beta\sigma_0 - \alpha\sigma_1)\,, \tag{2.4}$$

$$\alpha^2 + \beta^2 = 2\,, \tag{2.5}$$

where all other coordinates vanish ($\bar{x}^2 = \bar{x}^3 = \bar{y}^{s'} = 0$) and the last equation (2.5) follows from the conformal gauge condition. The original solution of [4] corresponds to $\alpha = \sqrt{2},\ \beta = 0$.

The boundary $z = 0$ is reached in the limit $\sigma_a \to \infty$ (assuming that $\sigma_a$ run in the infinite range and $\alpha, \beta \geq 0$). The induced metric is $ds_2^2 = d\sigma_0^2 + d\sigma_1^2$ so that the value of the classical action is simply

$$\bar{S}_E = \frac{\sqrt{\lambda}}{2\pi}V_2\,, \qquad V_2 = \int d^2\sigma\ . \tag{2.6}$$

For cusp anomaly interpretation it requires a regularization as discussed in [4, 29, 25]. This issue will not be important for us here as quantum corrections to (2.6) will also scale as $V_2$ and we will be interested in the value of the overall coefficient $f(\lambda)$ in (1.8).

The free parameters $\alpha, \beta$ included for generality, reflect the possibility of making SO(2) rotations of the world-sheet coordinates $\sigma_a$ which leave invariant the sigma model conformal-gauge equations of motion and constraints.

Under the 2d duality (T-duality) transformation $z^{-2}\partial_a x^m \to \epsilon_{ab}\partial^b\tilde{x}^m$ the solution (2.3)–(2.5) is essentially mapped into itself: the duality is equivalent to interchanging $\sigma_0 \to \sigma_1,\ \sigma_1 \to -\sigma_0$ and inverting $z$ which can be implemented by changing the signs of $\alpha, \beta$ and shiftinh $\sigma_a$ by constants. This is the reason why, instead of starting with the original $AdS_5 \times S^5$ action (containing quartic fermionic terms) and expanding the string path integral (giving the expectation value of the corresponding Wilson loop on the gauge theory side [27, 28]) near the null cusp solution [4] in order to extract from it the cusp anomaly coefficient $f(\lambda)$, we may formally start with the T-dual action (2.1) and expand

$AdS_5 \times S^5$ action has two different GS superstring extensions with different fermionic parts: the familiar one [23] corresponding to the near-core D3-brane background where $AdS_5 \times S^5$ space is supported by the RR 5-form flux and the T-dual one corresponding [31, 25] to the near-core smeared D-instanton background where the $AdS_5 \times S^5$ space is supported by the RR scalar and dilaton. For other potential applications of this T-dual action see [29, 25].

it near the equivalent null cusp solution in (2.3)–(2.5). As was already checked in [25], this procedure leads indeed to the same 1-loop coefficient $\mathrm{a}_1$ in (1.2) as found in the closed string approach (i.e. in the energy of the closed spinning string).

To expose the fact that the SO(2) 2d Euclidean rotational invariance of the conformal-gauge string sigma model[13] is only spontaneously broken by the classical background, it is useful to write the solution (2.3)–(2.5) in terms of the 2-vectors $n_{1a}, n_{2a}$ $(a = 0,1)$

$$n_1 = \frac{1}{\sqrt{2}}(\alpha,\beta)\,, \qquad n_2 = \frac{1}{\sqrt{2}}(-\beta,\alpha)\,, \qquad n_1\cdot n_1 = n_2\cdot n_2 = 1\,, \qquad n_1\cdot n_2 = 0\ , \quad (2.7)$$

i.e. $(n\cdot\sigma \equiv n^a\sigma_a)$

$$\bar z = \sqrt{2}\ e^{-\sqrt{2}\ n_1\cdot\sigma}\,, \qquad \bar x^0 \pm \bar x^1 = e^{-\sqrt{2}\ (n_1\cdot\sigma \pm\ n_2\cdot\sigma)}\ . \quad (2.8)$$

In particular, in the simple case of $\beta = 0$ we have

$$\alpha = \sqrt{2}\,, \qquad \beta = 0\,, \qquad n_1 = (1,0)\,, \qquad n_2 = (0,1)\ . \quad (2.9)$$

Below we will express the string Lagrangian for fluctuations near the null cusp solution in terms of the constant vectors $n_a$. This will help to make the structure of the quantum contributions more transparent.

## 2.2 Fluctuation lagrangian

To find the string fluctuation Lagrangian near the background (2.8) it is useful to utilize the observation of [25] that written in global $AdS_5$ coordinates it can be related (by an SO(2,4) isometry and an analytic continuation) to the scaling limit [3, 30] of the spinning closed string solution of [2]. The latter background is effectively homogeneous,[14] i.e. the corresponding fluctuation Lagrangian should have constant coefficients after an appropriate choice of basis of the fluctuation fields.[15]

[13] In general, the Euclidean classical string sigma model equations and conformal gauge constraints are covariant under the residual holomorphic conformal transformations of $\sigma_1 + i\sigma_0$ and $\sigma_1 - i\sigma_0$.

[14] One can make this explicit by an analytic continuation to an $S^5$ solution [30, 7] or directly by a special choice of coordinates in $AdS_5$ as discussed in [25, 26].

[15] To make the homogeneous nature of the solution (2.8) explicit it is useful to choose a different set of coordinates in the Poincare patch of $AdS_5$: $ds^2 = dr^2 + e^{-2r}dx^m dx_m = dr^2 + (dh^m + h^m dr)(dh_m + h_m dr)$ where $z = e^r$ and $h^m = \frac{x^m}{z} = e^{-r}x^m$ ($m = 0,1,2,3$ and the metric has signature $(-,+,+,+)$). Next, we set $h^\pm = h^0 \pm h^1 = ve^{\pm w}$. Then the $AdS_5$ metric above takes the form $ds^2 = dr^2 - (dv + vdr)^2 + v^2dw^2 + (dh_i + h_i dr)(dh_i + h_i dr)$, so that the shifts of $r, w$ are linear isometries $(i = 2,3)$. Let us assume that the world-sheet signature is euclidean and consider the corresponding string action in conformal gauge. Then simplest solution to look for is a homogeneous one where only the two isometric coordinates are non-zero and linear: $v = v_0 = \text{const},\ r = k_a\sigma_a,\ w = m_a\sigma_a,\quad h_i = 0$ ($k$ and $m$ are constant 2-vectors). Note that this ansatz makes sense only for an infinite open string since $r$ and $w$ are non-compact coordinates. The equations of motion are satisfied if $k^2 = m^2$ and the conformal gauge constraints give (assuming the induced metric has standard flat form): $(1 - v_0^2)k_a k_b + v_0^2 m_a m_b = \delta_{ab}$, i.e. $k^2 = m^2 = 2,\ k_a m_a = 0,\ v_0^2 = \frac{1}{2}$. This gives $z = e^{k\cdot\sigma},\ \ x^\pm = ve^r h^\pm = \frac{1}{\sqrt{2}}e^{(k\pm m)\cdot\sigma}$, i.e. brings us back to solution (2.8) after a trivial rescaling of $z$ and $x_m$ and renaming of the constant vectors. The fluctuation Lagrangian written in terms of $r, v, w, h_i$ has only constant coefficients and at most quartic vertices. Let us mention a generalization of the above solution to

In conformal gauge the $AdS_5$ and $S^5$ parts of the bosonic fluctuation Lagrangian are decoupled and can be written as (see also [25])

$$\tilde{L}_B = \frac{\sqrt{\lambda}}{4\pi}\mathcal{L}_B\,, \qquad \mathcal{L}_B = \mathcal{L}_{\rm AdS_5} + \mathcal{L}_{\rm S^5} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \dots\,, \tag{2.10}$$

$$\begin{aligned}\mathcal{L}_{2,\rm AdS_5} &= -(\partial\phi)^2 + \frac{1}{2}(\partial\varphi_1)^2 + \frac{1}{2}(\partial\varphi_2)^2 + 4\phi(n_2\cdot\partial\varphi_1 + n_1\cdot\partial\varphi_2) \\ &\quad + (\partial\xi)^2 + (\partial\eta)^2 + 2\xi^2 + 2\eta^2\,,\end{aligned} \tag{2.11}$$

$$\mathcal{L}_{3,\rm AdS_5} = \phi\big[(\partial\varphi_2)^2 - (\partial\varphi_1)^2\big] - 2(\xi^2+\eta^2)(n_2\cdot\partial\varphi_1 - n_1\cdot\partial\varphi_2)\,, \tag{2.12}$$

$$\mathcal{L}_{4,\rm AdS_5} = -\frac{8}{3}\phi^3(n_1\cdot\partial\varphi_2 + n_2\cdot\partial\varphi_1) + (\xi^2+\eta^2)\,\mathcal{L}_{2,\rm AdS_5} - 2\xi\eta\partial\xi\partial\eta\,, \tag{2.13}$$

$$\mathcal{L}_{2,\rm S^5} = \partial y^{s'}\partial y^{s'}\,, \qquad \mathcal{L}_{3,\rm S^5} = 0\,, \qquad \mathcal{L}_{4,\rm S^5} = -\frac{1}{2}y^2\,\partial y^{s'}\partial y^{s'}\,. \tag{2.14}$$

Here $\partial$ stands for $\partial_a$ and $n\cdot\partial = n_a\partial_a$, etc. The background dependence is represented by the constant 2-vectors $n_1$ and $n_2$ in (2.7). The fields $\phi, \varphi_1, \varphi_2, \xi, \eta$ are fluctuations in the five $AdS_5$ directions,[16] while $y^{s'}$ are $S^5$ coordinates from (2.2) that have zero background values. The massless time-like (ghost) fluctuation $\phi$ should eventually decouple together with another massless longitudinal mode (their trivial 1-loop contribution cancels against the decoupled conformal gauge ghost contribution).

The explicit relation between $\phi, \varphi_1, \varphi_2, \xi, \eta$ and the fluctuations of the original Poincare coordinates $z, x^m$ is given in appendix A. There we also present the resulting bosonic propagator which is non-diagonal in the $\phi, \varphi_1, \varphi_2$ directions.

Finding a convenient (constant coefficient) form for the quadratic fermionic term as well as for the fermion-boson coupling terms following from the action (2.1) requires us to perform a nontrivial rotation of fermions. This can be done in two steps. First, we note that the world sheet position dependence in the terms involving the coordinates transverse to the boundary directions arises entirely from the overall factor of $z$ in $z^s = z\hat{z}^s(y)$ (on the solution (2.8) we have $\bar{y}^{s'} = 0$, $\bar{\hat{z}}^{4,\dots,8} = 0$, $\bar{\hat{z}}^9 = 1$). Redefining $\theta \mapsto \theta/\sqrt{\bar{z}}$ and making use of the identity $\bar\theta\Gamma^A\theta = 0$ leads to the following expression for the fermionic term in the square brackets in (2.1)

$$\mathcal{L}_F = 4\,\epsilon^{ab}\,\bar\theta\left[\frac{\partial_a x^m}{\bar z}\Gamma_m \;+\; \left(\frac{\partial_a z}{\bar z}\hat z^s + \frac{z}{\bar z}\partial_a\hat z^s\right)\Gamma_s\right]\partial_b\theta\,. \tag{2.15}$$

Since $\bar z$ in (2.8) is exponential in $\sigma_a$ the terms with $\hat z^s$ in (2.15) will now have constant coefficients once expanded near the solution. A second local redefinition of $\theta$ is needed in order to take into account that $x^0$ and $x^1$ have nontrivial backgrounds in (2.8). In general, the background value $\bar N$ of

$$N^u_a \equiv \frac{\partial_a x^u}{\bar z}\,, \qquad\qquad u = 0,1;\quad a = 0,1 \tag{2.16}$$

the case when there is also a "rotation" in $S^5$ direction. The solution related to the scaling limit [30] of $(S,J)$ string [3] in the same way as described in [25] has also a non-trivial angle $\varphi = \nu'\sigma_0$ of $S^5$ ($\nu' = i\nu$, $J = \sqrt{\lambda}\nu$ in Minkowski signature) and $v = v_0 = \frac{1}{\sqrt{2}}$, $r = -\kappa\sigma_0 + \mu\sigma_1 + \frac{1}{2}\ln 2$, $w = \kappa\sigma_0 + \mu\sigma_1$, $\kappa^2 = \mu^2 - \nu'^2$. Equivalently, $z = \sqrt{2}e^{-\kappa\sigma_0+\mu\sigma_1}$, $x^+ = e^{2\mu\sigma_1}$, $x^- = e^{-2\kappa\sigma_0}$. The conformal factor of flat induced metric is equal to 1 when $\mu = 1$.

[16] The fields $\xi, \eta$ are related to fluctuations of $x^2, x^3$ in (2.1) that are zero in the solution (2.8).



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is not an SO$(1,1)$ rotation matrix:

$$\bar{N}^u_a \bar{N}^v_b \eta_{uv} = n_{2a} n_{2b} - n_{1a} n_{1b} \ . \tag{2.17}$$

It is nevertheless possible (though somewhat complicated) to find an SO$(1,1)$ rotation of fermions that removes the position dependence from their action. For simplicity, it is sufficient to consider the case of $\beta = 0$ in (2.9). Then we get

$$\bar{N}^u_a \bar{N}^v_b \eta_{uv} = \eta_{ab} \,, \tag{2.18}$$

and thus the required $\sigma_a$ dependent rotation of $\theta$ is

$$\theta \quad \mapsto \quad \left[ \cosh\left( \frac{1}{\sqrt{2}}\, n_2 \cdot \sigma \right) + \sinh\left( \frac{1}{\sqrt{2}}\, n_2 \cdot \sigma \right) \Gamma^0 \Gamma^1 \right] \theta \ . \tag{2.19}$$

Moreover, it turns out that the matrix

$$\mathcal{N}_{ab} = N^u_a \bar{N}^v_b \eta_{uv} \tag{2.20}$$

expanded near the classical solution has only terms with constant coefficients in front of the bosonic fluctuations (its expression to leading order in bosonic fluctuations is given in appendix A).[17]

Taking into account the effect of the rotation (2.19) and making further use of the identity $\bar{\theta}\Gamma^A \theta = 0$, we finally find for the fermionic part of the fluctuation Lagrangian $(a,b = 0,1;\ i,j = 2,3;\ s,t = 4,5,\dots,9)$:

$$\begin{aligned} \mathcal{L}_F = {} & 4\epsilon^{ab}\, \bar{\theta} \Big[ -\mathcal{N}_{ac}\Gamma^c + \frac{\partial_a x^i}{\bar{z}}\Gamma_i + \left( \frac{\partial_a z}{\bar{z}} \hat{z}^s + \frac{z}{\bar{z}} \partial_a \hat{z}^s \right) \Gamma_s \Big] \partial_b \theta \\ & - 2\sqrt{2}\ \epsilon^{ab}\ n_{2b}\ \bar{\theta} \Big[ \frac{\partial_a x^i}{\bar{z}}\Gamma_i + \left( \frac{\partial_a z}{\bar{z}} \hat{z}^s + \frac{z}{\bar{z}} \partial_a \hat{z}^s \right) \Gamma_s \Big] \Gamma^0 \Gamma^1 \theta \ . \end{aligned} \tag{2.21}$$

The second line appears due to the rotation (2.19). The bosonic fields here can be expanded in fluctuations $\phi, \varphi_1, \varphi_2, \xi, \eta$ and $y^{s'}$ (using the relations in appendix A) leading to fermion-fermion-boson and fermion-fermion-boson-boson quantum vertices needed to compute the 2-loop diagrams in figure 1.

The quadratic term in (2.21) determining a non-degenerate fermionic propagator can be written as

$$\mathcal{L}_{2F} = 2\sqrt{2}\ \epsilon^{ab}\ \bar{\theta} \Big[ (-\Gamma^0 + \sqrt{2}\Gamma^9) n_{1a} \partial_b - \Gamma^1 n_{2a} \partial_b + n_{1a} n_{2b} \Gamma^{019} \Big] \theta \ . \tag{2.22}$$

The $\Gamma^{019}$ term produces a non-zero mass (equal to 1) for the 8 independent fermionic fluctuations (see also [25]).

## 3. Quantum corrections

Let us now turn to the computation of quantum loop corrections to the effective action as defined by path integral based on the action given by the sum of the bosonic (2.10) and the fermionic (2.21) parts.

[17] In the general case of $\beta \neq 0$ when $\bar{N}^u_a \bar{N}^v_b \eta_{uv}$ is still a constant off-diagonal matrix the expansion of $N^u_a \bar{N}^v_b \eta_{uv}$ around the classical solution has again the constant coefficients.



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### 3.1 One-loop contribution

Let us start with reviewing the 1-loop result [3, 25]. From the quadratic part of the above fluctuation Lagrangian it is straightforward to recover the mass spectrum and the 1-loop value of the effective action (1.8) and thus the 1-loop coefficient in the cusp anomaly function.

Extracting the bosonic kinetic operator from (2.10) and computing its determinant in 2d momentum representation (the propagator $K_B^{-1}(q)$ is given in appendix A) we find

$$\det K_B(q) = -2^8\,(q^2)^7\,(q^2+2)^2(q^2+4)\ . \tag{3.1}$$

This means that the bosonic spectrum contains seven massless scalars, two scalars with mass $\sqrt{2}$ and one scalar with mass 2.

Performing a similar computation of the fermionic spectrum from the determinant of the fermionic kinetic operator in (2.22) we get ($n\times q \equiv \epsilon^{ab}n_a q_b$; see (2.9))

$$\det K_F(q) = \left[(n_1\times n_2)^2 + (n_1\times q)^2 + (n_2\times q)^2\right]^8 = 2^{16}(q^2+1)^8\,, \tag{3.2}$$

implying that the spectrum contains eight fermions with mass 1.

This coincides with the spectrum of fluctuations around the folded spinning string [3], as was already discussed in [25]. Taking into account that the conformal-gauge ghost contribution cancels the contribution of the two bosonic massless modes, the 1-loop effective action is found to be given by the same expression as in [3, 30, 25]

$$\Gamma_1 = \frac{1}{2}V_2\int\frac{d^2q}{(2\pi)^2}\Big[\ln(q^2+4)+2\ln(q^2+2)+5\ln q^2-8\ln(q^2+1)\Big] = -\frac{3\ln 2}{2\pi}V_2\ . \tag{3.3}$$

This leads (using (1.8)) to the value of $a_1$ in (1.2).[18]

### 3.2 Two-loop contribution

The 1-loop result (3.3) is manifestly finite: the logarithmic UV divergences cancel between the bosonic and the fermionic terms. As was discussed in detail in [7], the issue of potential higher-loop UV divergences in Green-Schwarz action expanded near a particular string background is subtle, due in particular to its lack of manifest power counting renormalizability.[19]

To get rid of power divergences in [7] we attempted to use dimensional regularization (as is common in the treatment of 2d sigma models). Continuing the $AdS_5\times S^5$ GS action to $d = 2-2\epsilon$ dimensions appears, however, to be inconsistent as this spoils its classical $\kappa$ symmetry.[20] While the regularization procedure used in [7] made possible to find the

[18] Introducing a UV cutoff in the momentum integral in (3.3) one finds that the finite part proportional to $-3\ln 2$ comes only from the bosonic mode contribution while the role of the fermion contribution is to cancel the bosonic UV divergence.

[19] As for the 2d IR divergences, they cancel in on-shell effective action as expected on general grounds [7].

[20] Ideally, the regulator should be introduced before gauge fixing so that it preserves all local invariances (and as many of the global invariances as possible). The presence of the Levi-Civita tensor (WZ) term in the $AdS_5\times S^5$ GS action and the related 2d self-duality property of the $\kappa$-symmetry parameters makes dimensional continuation problematic.



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non-trivial finite part of the 2-loop effective action and reproduce the value (1.3) of $\mathrm{a}_2$,[21] it did not allow us to check the expected cancellation of the logarithmic divergences.

Here we resolve this problem. A consistent computational procedure appears to be as follows. One should first not use any explicit regularization and rearrange the momentum integrals (directly in $d = 2$) to extract all potential logarithmically divergent contributions.[22] Remarkably, by direct computation of the 2-loop graphs in figure 1 starting with the action (2.1), (2.10), (2.21) we have found that the thus extracted $\ln\Lambda$ and $\ln\Lambda^2$ logarithmic divergences *cancel separately* in the sum of purely bosonic graphs (figure 1a, 1b) and the sum of graphs with fermionic propagators (figure 1c, 1d). The remaining power divergent terms can then be regularized away by a kind of analytic regularization prescription. In fact, they should cancel against the invariant measure and $\kappa$-symmetry ghost contributions in a systematic treatment that takes into account all local $\delta^{(2)}(0)$ contributions.

A variant of such regularization procedure is a version of "dimensional regularization" that was found in [7] to preserve the BPS nature ($Z = 1$) of the expansion near the BMN point-like string at 2-loops (see appendix C in [7]). It assumes that the use of all algebraic manipulations with momentum integrals as well as of symmetric integration identities is done strictly in two dimensions. The resulting 2d Lorentz *covariant* integrals are then continued to $d = 2 - 2\epsilon$ as a way to get rid of power divergences. It turns out that the simple poles in $\frac{1}{\epsilon}$ then cancel at the same time as the double poles, and that happens separately for the bosonic and the fermionic contributions.

This prescription amounts to a consistent definition (respecting all relevant symmetries of the classical action) of the $AdS_5 \times S^5$ string theory as a 2d quantum conformal theory. As we find below, the resulting 2-loop effective action is then finite and reproduces the value in (1.3).

Before turning to the summary of our 2-loop results let us comment some more on the cancellation of the logarithmic UV divergences. As was pointed out above, while at the 1-loop order the logarithmic UV divergences were cancelling *between* the bosonic and the fermionic contributions the 2-loop cancellation pattern is different: the logarithmic UV divergences cancel separately in the bosonic and fermionic graph contributions. This may seem surprising being in an apparent contradiction with the well-known expression for the 2-loop $\beta$-function for a generic bosonic sigma model (found in dimensional regularization with minimal subtraction [34]), $\beta_{\mu\nu} = R_{\mu\nu} + \frac{\alpha'}{2} R_{\mu\lambda\rho\sigma} R_\nu{}^{\lambda\rho\sigma} + \ldots$. The point, however, is that the form of the 2-loop correction is renormalization scheme dependent (in a generic scheme it contains other $R_{\mu\nu}$ dependent terms, see, e.g., [35]) and for particular geometries (e.g., having zero Weyl tensor) one may achieve cancellation of the 2-loop correction in a special

[21] The value of $\mathrm{a}_2$ in the original version of [7] was incorrect: (i) the cancellation of the second transcendental constant $\tilde{\mathrm{K}}$ in the bosonic contribution was overlooked; (ii) the normalization of the fermionic contribution was off by factor of 2; (iii) the computation was done for an $S^5$ background related to the relevant $AdS_5$ rotating string background by an analytic continuation [37, 30] that also inverts the string tension, so that the result of the 2-loop $S^5$ computation should be taken at the end with an opposite sign. These errors were corrected in the revised version (v4) of [7].

[22] The procedure of required rearranging of the momentum integrals by reducing the power of momenta in the numerators was described in detail in [7].

scheme.[23] Moreover, for the homogeneous spaces like $AdS_5$ and $S^5$ there is no $\frac{1}{\epsilon^2} \sim \ln^2 \Lambda$ 2-loop UV divergences (which are in general proportional to covariant derivatives of $R_{\mu\nu}$).

In fact, starting formally with such a sigma model defined in $d$ dimensions one finds [34, 7] that the potentially divergent contribution is proportional to $d - 2 = -2\epsilon$ times the square of the tadpole integral $I[m] = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+m^2}$ (so that $\frac{1}{\epsilon^2}$ pole cancels out). Then if one uses a scheme in which one first combines the contributions of momentum integrals directly in $d = 2$ then all logarithmic divergences cancel out. In this natural regularization prescription the bosonic $AdS_5 \times S^5$ sigma model defined directly in $d = 2$ is manifestly 2-loop finite.[24] A non-trivial check of the quantum consistency of the $AdS_5 \times S^5$ action is that the same applies separately also to the fermionic graph contribution. This is indeed what we have found by direct calculation starting with the action (2.1).

Given the mass spectrum of bosonic and fermionic fluctuations described above one may anticipate which momentum integrals may in principle appear in the the 2-loop effective action given by the sum of graphs in figure 1. The values of masses are $0, \sqrt{2}, 2$ (bosonic) and 1 (fermionic) but not all combinations of masses in the propagators actually happen to appear in the final result. Let us define

$$I[m_1, m_2, m_3] = \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^4} \frac{\delta^{(2)}(q_1+q_2+q_3)}{(q_1^2+m_1^2)(q_2^2+m_2^2)(q_3^2+m_3^2)}\,, \tag{3.4}$$

$$I[m_1, m_2] = \int \frac{d^2q_1 d^2q_2 d^2q_3}{(2\pi)^4} \frac{\delta^{(2)}(q_1+q_2+q_3)}{(q_1^2+m_1^2)(q_2^2+m_2^2)} = I[m_1]I[m_2]\,, \tag{3.5}$$

$$I[m] = \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2+m^2}\,. \tag{3.6}$$

The integral (3.4) is UV-finite while (3.5) exhibits $\ln^2 \Lambda$ and $\ln \Lambda$ UV divergences. The momentum integrals that appear in the direct computation of 2-loop graphs starting with the action (2.10), (2.21) can be expressed in terms of the sum of integrals of the above type plus the power divergent contributions proportional to the square of $I_0 = \int \frac{d^2q}{(2\pi)^2}$ and to $I_0 I[m]$; these we set to zero by an analytic (e.g. dimensional) regularization [40].

The explicit calculation has shown that only two special cases of the finite integral (3.4)

[23] For example, for a sphere sigma model there is only one coupling parameter (its radius) and thus the 2-loop $\beta$ function coefficient can be made zero by a coupling redefinition. Equivalently, the general-scheme expression for 2-loop term in $\beta_{\mu\nu}$ involves terms obtained from the 1-loop term by a coupling redefinition $G'_{\mu\nu} = G_{\mu\nu} + \alpha'(b_1 R_{\mu\nu} + b_2 G_{\mu\nu} R) + \ldots$ and in the case when the Weyl tensor vanishes they can cancel the $R_{\mu\lambda\rho\sigma} R_\nu{}^{\lambda\rho\sigma}$ term. Note also that the standard lore that the first two coefficients in the beta-function are scheme independent in 1-coupling theory does not apply to 2d sigma models, including the $S^n$ one, since here the scaling of the two-loop correction with the coupling is different than, say in gauge theory (which in turn is related to the 2-derivative nature of the sigma model vertices). A scheme in which 2-loop and higher terms in the $\beta$-function depend only on the Weyl tensor and thus vanish for $AdS_n$ or $S^n$ case was discussed also, e.g., in [36].

[24] The covariant local measure contribution cancels also power divergences [7].



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remain in the final answer.[25] They are [7]

$$I[\sqrt{2},\sqrt{2},2]=\frac{\mathrm{K}}{(4\pi)^2}\,,\qquad I[1,1,\sqrt{2}]=\frac{2\mathrm{K}}{(4\pi)^2}\,,\qquad \mathrm{K}\equiv\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\ . \tag{3.7}$$

In general, the contribution of bosonic graphs to the 2-loop effective action in a theory with cubic and quartic vertices is

$$\Gamma_{2B}=\frac{4\pi}{\sqrt{\lambda}}\int d^2\sigma\Big(-\frac{1}{12}A_{3B}+\frac{1}{8}A_{4B}\Big)\,, \tag{3.8}$$

where $A_{3B}$ and $A_{4B}$ are, respectively the contributions of the graphs with topologies shown in figure 1a and 1b. In the case of the bosonic $AdS_5\times S^5$ sigma model computed according to the regularization prescription described above they turn out to be

$$A_{3B} = 12\ (I[\sqrt{2},\sqrt{2}]+\ 4\ I[2,2])-\ 24\ I[\sqrt{2},\sqrt{2},2]\,, \tag{3.9}$$

$$A_{4B} = 8\ (I[\sqrt{2},\sqrt{2}]+\ 4\ I[2,2])\ . \tag{3.10}$$

Here $I[\sqrt{2},\sqrt{2}]$ and $I[2,2]$ are the $\ln^2\Lambda$ and $\ln\Lambda$ UV divergent integrals (3.5). Combining them in (3.8) one finds that they *cancel* leaving us with a finite result proportional to the Catalan's constant (see (3.7))

$$\Gamma_{2B}=\frac{4\pi}{\sqrt{\lambda}}\frac{2\mathrm{K}}{(4\pi)^2}V_2\ . \tag{3.11}$$

Similarly, the contribution of 2-loop graphs with fermion propagators in a theory with fermion-fermion-boson and fermion-fermion-boson-boson couplings is in general

$$\Gamma_{2F}\ =\frac{4\pi}{\sqrt{\lambda}}\int d^2\sigma\Big(\frac{1}{16}A_{3F}+\frac{1}{8}A_{4F}\Big)\,, \tag{3.12}$$

where $A_{3F}$ and $A_{4F}$ are produced, respectively, by graphs with topologies in figure 1c and 1d. The explicit calculation yields finite results

$$A_{3F}\ =-32\ I[1,1,\sqrt{2}]\,,\qquad\qquad A_{4F}=0\,, \tag{3.13}$$

so that

$$\Gamma_{2F}=-\frac{4\pi}{\sqrt{\lambda}}\frac{4\mathrm{K}}{(4\pi)^2}V_2\ . \tag{3.14}$$

As was already mentioned above, the cancellation of divergences in the sum of graphs with fermion propagators represents a strong consistency test of the quantum GS action. The absence of UV divergences *separately* in graphs in figures 1c and 1d potentially suggests the existence of additional non-manifest symmetries in the fermionic action that are preserved by the $\kappa$-symmetry gauge as well as by our regularization prescription.

[25] All possible combinations of masses occur at the intermediate steps. As in [7] the calculations were done using `Mathematica`-based computer program.



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Combining the bosonic (3.11) and the fermionc (3.14) contributions, we finish with

$$\Gamma_2 = \Gamma_{2B} + \Gamma_{2F} = \frac{\mathrm{a}_2}{2\pi\sqrt{\lambda}} V_2 \, , \tag{3.15}$$

$$\mathrm{a}_2 = \mathrm{a}_{2B} + \mathrm{a}_{2F} = \mathrm{K} - 2\mathrm{K} = -\mathrm{K} \, . \tag{3.16}$$

Similar results for the finite parts of $\Gamma_{2B}$ and $\Gamma_{2F}$ were found also in an independent computation for a closely related $S^5$ background in [7]. The overall sign of the result for $\Gamma_2$ was, however, opposite. This is consistent with the equivalence of the two $AdS_5$ and $S^5$ solutions since the analytic continuation relating them implies that one should also change the sign of the string tension $\sqrt{\lambda} \to -\sqrt{\lambda}$, i.e. to reverse the sign of all even-loop terms in the effective action. Formally, this reverses the sign of the coefficient $\mathrm{a}_2$ in $\Gamma_2$, leading again to the result in (3.16).

### 3.3 Higher-loop contributions

Going to higher, e.g. 3-loop, order is in principle straightforward. We again expect that all logarithmic divergences will cancel directly in $d = 2$ while power divergences can be unambiguously separated and regularized away.

Based on the spectrum of fluctuations and the form of the propagators and vertices in the string fluctuation action it is relatively straightforward to determine the general structure of the finite higher loop contributions to the effective action and thus to the strong-coupling expansion of cusp anomaly function in (1.1), (1.8) as predicted by the string inverse tension expansion.

On dimensional grounds, the finite contribution to the effective action or cusp anomaly comes from momentum integrals of mass dimension $-2$ (cf. (3.3), (3.4)). Most vertices in the action (2.1) contain derivatives; employing partial fractioning and 2d Lorentz invariance these derivatives may be used to cancel some of the propagators. Since many of the fluctuation fields in the theory are massive, this leaves behind terms with uncanceled propagators and with the momenta in the numerators replaced by the mass values. Thus, the $L$-loop contribution to the effective action can be expressed in terms of *scalar* vacuum integrals whose topology is that of the initial Feynman diagrams as well as that of the "daughter" diagrams obtained by collapsing some of the propagators.

At each loop order there exists a new set of "maximally irreducible" topologies (see, e.g., figures 1a and 1c at $L = 2$ and figures 2b, 2e,2g and 2h at $L = 3$). At $L$ loops these topologies contain at most $(L + 1)$-point vertices. All other topologies that can appear at an $L$-loop order are inherited from lower loop orders by simply adjoining lower loop graphs in such a way that the total number of loops is $L$ (see figures 1b, 1d and figures 2a,2c, 2d, and 2f).[26]

The number of irreducible sigma model diagrams grows factorially with $L$ (there are more graphs than, say in $\phi^4$ theory). A type of graph that potentially occurs at each loop order is shown in figure 3. It may contain various combinations of propagators with

[26]Since the tadpole integral (3.6) is logarithmically divergent and assuming that the finiteness of the partition function persists to all loop orders, such graphs will either not appear at all or they will involve both the bosonic and the fermionic propagators in such a way that the sum of all of them is finite.



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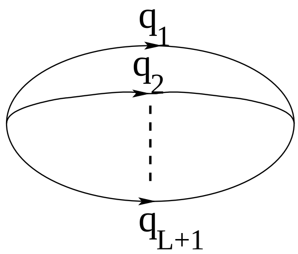


**Figure 3:** $L$-loop "maximally irreducible" sunset-type graph

mass values $(1, \sqrt{2}, 2)$ from the spectrum of our theory (massless propagators should not appear at the end as all IR divergences should cancel out). On general grounds, most of such integrals may contribute to the the effective action;[27] explicit calculations are then necessary to fix their coefficients. Some of these coefficients may, in fact, vanish, as was, for example, the case with $I[2,2,2]$ in the bosonic part (3.8) and with $I[1,1,2]$ in the fermionic part of the 2-loop result (see also [7]).

To examine which new transcendental numbers may possibly appear as coefficients in $\Gamma$ at the 3-loop order let us consider a generalization of the integral (3.4)

$$I[m_1,m_2,m_3,m_4] = \int \frac{d^2q_1 d^2q_2 d^2q_3 d^3q_4}{(2\pi)^8} \frac{\delta^{(2)}(q_1+q_2+q_3+q_4)}{(q_1^2+m_1^2)(q_2^2+m_2^2)(q_3^2+m_3^2)(q_4^2+m_4^2)}\,, \quad (3.17)$$

which corresponds to a graph of "maximally irreducible" topology shown in figure 3 with $L = 3$ (i.e. to figure 2b). It is not easy to compute this integral for arbitrary values of the masses $m_k$. Using the values of $m_k$ that appear in our spectrum of fluctuations and making a simplifying assumption that the four masses split into two equal pairs, we find:[28]

$$I[\sqrt{2},\sqrt{2},\sqrt{2},\sqrt{2}] = 2I[2,2,2,2] = \frac{7\,\zeta(3)}{2(4\pi)^4}\,, \quad (3.18)$$

$$\begin{aligned} I[\sqrt{2},2,\sqrt{2},2] = \frac{1}{2}I[1,\sqrt{2},1,\sqrt{2}] = \frac{1}{4\sqrt{2}(4\pi)^4}\Bigg[&\ln(3+2\sqrt{2})\ (\ln 2)^2 \\ &+4\sqrt{2}\left[\mathrm{Li}_2\Big(\frac{1}{\sqrt{2}}\Big)-\mathrm{Li}_2\Big(-\frac{1}{\sqrt{2}}\Big)\right]+8\left[\mathrm{Li}_3\Big(\frac{1}{\sqrt{2}}\Big)-\mathrm{Li}_3\Big(-\frac{1}{\sqrt{2}}\Big)\right]\Bigg], \end{aligned} \quad (3.19)$$

$$\begin{aligned} I[1,2,1,2] = \frac{1}{(4\pi)^4}\Bigg[&\ln 3\ (\ln 2)^2 \\ &+2\ln 2\left[\mathrm{Li}_2\Big(\frac{1}{2}\Big)-\mathrm{Li}_2\Big(-\frac{1}{2}\Big)\right]+2\left[\mathrm{Li}_3\Big(\frac{1}{2}\Big)-\mathrm{Li}_3\Big(-\frac{1}{2}\Big)\right]\Bigg]. \end{aligned} \quad (3.20)$$

As follows from (3.18), the $\zeta(3)$ coefficient makes a natural appearance at the 3-loop order in the superstring sigma model partition function.[29] It is therefore very likely that it will

[27] Some of them may be ruled out by taking into account the structure of possible vertices in the superstring action (e.g., $I[1,1,1]$ with 3 fermionic masses is obviously not allowed).

[28] The first integral here is related to the one found in [41]. It is interesting to note that integrals with irrational mass ratios do not have definite transcendentality.

[29] As is well known, it also appears in the 4-loop sigma model $\beta$-function (computed in a mininimal subtraction scheme) in the case of a generic target space metric [38, 39].

be present in the expression for the 3-loop coefficient $\mathrm{a}_3$ in the cusp anomaly function. This observation suggests a superstring (2d Feynman-diagrammatic) interpretation to the value of the coefficient $\mathrm{a}_3$ in (1.4) found from the BES equation in [18].

Integrals for higher loop maximally irreducible graphs in figure 3 are harder to evaluate. Their general expression written in 2d coordinate space is

$$I[m_1, m_2, \ldots, m_{L+1}] = \int d^2x \prod_{i=1}^{L+1} \left[\frac{1}{2\pi} K_0(m_i|x|)\right], \tag{3.21}$$

where $K_0$ is the Bessel function ($\frac{1}{2\pi}K_0$ is the 2d massive scalar propagator). It would be interesting to relate their values[30] to the constants appearing in the expressions for higher-order $\mathrm{a}_L$ coefficients found in [18]: the odd-loop coefficient should start with zeta function $\zeta(L)$ and the even-loop one should start with the Dirichlet beta function $\beta(L)$ (cf. (1.3)–(1.6)).[31,32]

An underlying reason for this relation may be that the coordinate space representation of the 2d diagrams is given by integrals of products of the Bessel function $K_0$ and its derivatives while the integrals of Bessel functions appear also in the BES equation [6] and its strong-coupling solution in [18].

## 4. Concluding remarks

One consequence of the strong-coupling solution of the BES equation found in [18] was that the coefficients $\mathrm{a}_1, \mathrm{a}_2, \ldots$ are all negative and grow factorially. The series in (1.1) is then not Borel summable, i.e. its summation is ambiguous and this might be suggesting adding to (1.1) exponentially small terms $\sim e^{-k\sqrt{\lambda}}$ for some positive $k$.

By formally changing the sign of $\sqrt{\lambda}$

$$\sqrt{\lambda} \quad \to \quad -\sqrt{\lambda} \tag{4.1}$$

one finds that (1.1) becomes a sign-alternating and thus Borel summable series. This is puzzling since the *weak-coupling* expansion $f(\lambda) = b_1\lambda + b_2\lambda^2 + \ldots$ which is also described by the BES equation (and which has finite radius of convergence) is formally invariant under the sign change (4.1) and thus is "not aware" of the problem with summation of the strong-coupling expansion.

The string theory interpretation of $f(\lambda)$ as a coefficient in the partition function expanded near a perturbatively stable string solution would also suggest a standard asymptotic but Borel-summable expansion in $\frac{1}{\sqrt{\lambda}}$. However, the string theory result for $\mathrm{a}_2$ in

[30] Such sunset diagram integrals in different dimensions were extensively discussed in the literature (see [42] and references therein) and can be found numerically, but their analytic form is apparently not known beyond few simple examples.

[31] By writing a generating function for the equal mass sunset diagrams $F(t) = \int d^2x\ K_0(|x|)\ e^{-tK_0(|x|)}$ and approximating the Bessel function in the exponent as $K_0(|x|) \sim -\ln|x|$ one may see that at $L$ loops we expect a $\zeta(L)$-type transcendentality. We thank A. Pivovarov for this comment.

[32] In this connection let us mention the following useful relations: $\beta(n) = \frac{(-1)^n}{4^n(n-1)!}[\psi_{n-1}(\frac{1}{4}) - \psi_{n-1}(\frac{3}{4})]$, $\beta(2n) \propto \mathrm{Li}_{2n-1}(\frac{1}{4}) - \mathrm{Li}_{2n-1}(\frac{3}{4})$, $\beta(2n-1) \propto \pi^{2n-1}$, where $\psi_{n-1}(x)$ is the $n$-th derivative of $\ln\Gamma(x)$. They express the Dirichlet beta function in terms of quantities that naturally appear in loop integrals.



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$f(\lambda)$ found in [7] and here reproduces the negative sign of $\mathrm{a}_2$ in (1.3) and thus appears to support the conclusion of [18] about the lack of Borel-summability of the strong coupling expansion of $f(\lambda)$.[33]

According to standard discussions of the appearance of similar asymptotic series in QM and QFT problems this seems to suggest that the string background we are expanding about is actually unstable, despite its apparent stability under small fluctuations of string coordinates (all fluctuation modes in section 3 had non-negative values of squares of their masses).[34] The instability should manifest itself in the existence of complex energies but it is not obvious what might be the origin of this instability,

Absence of Borel summability of perturbative expansions occurs in all quantum-mechanical potential problems in which the expansion is done near a local (but not global) minimum of the potential [44, 47].[35] By analogy with such systems we may interpret this apparent non-summability of the string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ perturbation theory as a signal that the closed spinning string (or, equivalently, the null cusp solution) is only a local minimum of the $AdS_5 \times S^5$ superstring action. If this is indeed the correct interpretation, there should be a tunneling solution connecting it to some "global vacuum" state.

It is interesting to note that the details of our 1-loop and 2-loop calculations (cf. (3.3), (3.16)) suggest that the perturbation theory of the *bosonic* $AdS_5 \times S^5$ sigma model leads to a sign-alternating — and hence Borel-summable — series while the addition of the fermionic contributions spoils this feature.[36]

A non-resummable perturbation theory expansion is ambiguous in the sense that either the function being expanded has indeed a cut along the real axis in the coupling plane (in which case the expansion is meaningless) or the singularity is cancelled by terms whose derivatives vanish at the expansion point [45]. What happens in the present case, both from the Bethe ansatz and the string theory points of view, remains to be clarified.

One generalization of the 2-loop superstring computation described in this paper is to the case of null cusp solution with non-zero angular momentum $J$ in $S^5$. This in particular

[33] As was already mentioned above, the direct result of the computation in [7] was actually the opposite sign for $\mathrm{a}_2$. However, this computation was done for a complex (but perturbatively stable) $S^5$ solution related to the scaling limit [30] of the spinning string solution in $AdS_5$ by a formal complex automorphism of the $AdS_5 \times S^5$ string action which is an equivalence transformation provided one also inverts the sign of the string tension, i.e. of $\sqrt{\lambda}$. This effectively inverts the sign of $\mathrm{a}_2$.

[34] The standard argument [43] based on $e^2 \to -e^2$ continuation that makes perturbative vacuum unstable implies that the complex coupling constant space exhibits a cut on the negative real axis and consequently the expansion in small $e$ near a perturbatively stable vacuum should lead to an asymptotic sign-alternating and thus Borel summable series. In the case when the coefficients in the series are not sign-alternating, the Borel transformed series no longer converges, i.e. the perturbation series is not Borel-summable. From the standpoint of the argument of [43] this case appears to correspond to developing perturbation theory around an unstable vacuum state.

[35] In QFT context, ref. [46] argued that the large order behavior of coefficients of perturbation theory around a stable vacuum state may be explained by the existence of a classical euclidean solution of the effective action appearing at each order of perturbative expansion (a complex instanton). In [45] it was argued that the lack of Borel summability of perturbation theory near an unstable vacuum may be related to the existence of a "real instanton" of the original action.

[36] This might be an indication that the "global vacuum" may involve some nontrivial classical profiles for the fermionic fields.



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may allow one to verify the suggestion [26] that certain terms in the corresponding anomalous dimension found in the limit when $\frac{J}{\sqrt{\lambda}\ln S} \ll 1$ are determined only by the bosonic $S^5$ contributions. The relevant solution with non-zero $J$ is (cf. (2.8), (2.9), [25]; see also footnote 15 above)

$$\begin{aligned}
\bar{z} &= \sqrt{2}e^{-\kappa\,\sigma_0+\sigma_1}\,, \qquad\qquad \varphi = \nu'\,\sigma_0\,, \qquad\qquad \kappa = \sqrt{1-\nu'^2}\,,\\
\bar{x}^0 &= e^{-\kappa\,\sigma_0+\sigma_1}\cosh(\sigma_1+\kappa\sigma_0)\,, \qquad\qquad \bar{x}^1 = e^{-\kappa\,\sigma_0+\sigma_1}\sinh(\sigma_1+\kappa\sigma_0)\ .
\end{aligned} \tag{4.2}$$

Here $\varphi$ is an angle of $S^5$ and $\nu' = i\nu$, where $J = \sqrt{\lambda}\nu$ is the angular momentum of the corresponding spinning string background with Minkowski world sheet. A technical complication is that, unlike the case of $\nu' = 0$ discussed above, the denominator of the bosonic propagator no longer has a Lorentz-covariant form. Consequently, the direct calculation of momentum integrals becomes quite cumbersome. Moreover, the existence of fluctuation fields of mass proportional to $\nu'$ requires that their contribution is treated exactly. We leave this problem for the future.


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## A. Details of fluctuation Lagrangian

The relation between the Poincare coordinate fields $z, x^m$ and the fluctuation fields in (2.10) is as follows ($\epsilon$ is a formal expansion parameter that should be set to 1 at the end)

$$\begin{aligned}
z &= \frac{e^{-\sqrt{2}\ n_1\cdot\sigma-\epsilon\varphi_2}}{\sin(\frac{\pi}{4}+\epsilon\phi))\sqrt{1+\epsilon^2(\xi^2+\eta^2)}}\,,\\
x^0 &= e^{-\sqrt{2}\ n_1\cdot\sigma-\epsilon\varphi_2}\cosh(\sqrt{2}\ n_2\cdot\sigma-\epsilon\varphi_1)\cot\left(\frac{\pi}{4}+\epsilon\phi\right),\\
x^1 &= -e^{-\sqrt{2}\ n_1\cdot\sigma-\epsilon\varphi_2}\sinh(\sqrt{2}\ n_2\cdot\sigma-\epsilon\varphi_1)\cot\left(\frac{\pi}{4}+\epsilon\phi\right)\\
x^2 &= \frac{\epsilon\ e^{-\sqrt{2}\ n_1\cdot\sigma-\epsilon\varphi_2}\ \eta}{\sin(\frac{\pi}{4}+\epsilon\phi)\sqrt{1+\epsilon^2(\xi^2+\eta^2)}}\,,\\
x^3 &= \frac{\epsilon\ e^{-\sqrt{2}\ n_1\cdot\sigma-\epsilon\varphi_2}\ \xi}{\sin(\frac{\pi}{4}+\epsilon\phi)\sqrt{1+\epsilon^2(\xi^2+\eta^2)}}\ .
\end{aligned} \tag{A.1}$$

The expansion of the matrix (2.20), (2.16) that enters the action (2.21) has the form

$$\mathcal{N}_{ab}=\eta_{ab}+\epsilon\begin{pmatrix} 2\phi+\varphi_2-\frac{1}{\sqrt{2}}(n_1\cdot\partial\phi+n_1\cdot\partial\varphi_2) & \varphi_1-\frac{1}{\sqrt{2}}n_1\cdot\partial\varphi_1 \\ -\varphi_1-\sqrt{2}n_2\cdot\partial\phi-\frac{1}{\sqrt{2}}n_2\cdot\partial\varphi_2 & -2\phi-\varphi_2-\frac{1}{\sqrt{2}}n_2\cdot\partial\varphi_1 \end{pmatrix}_{ab}+\mathcal{O}(\epsilon^2). \quad \text{(A.2)}$$

The bosonic propagator corresponding to the quadratic part of the Lagrangian (2.10) is

$$K_B^{-1}(q)=\begin{pmatrix} \frac{-1}{2(q^2+4)} & \frac{i\sqrt{2}\ n_2\cdot q}{q^2\,(q^2+4)} & \frac{i\sqrt{2}\ n_1\cdot q}{q^2\,(q^2+4)} & 0 & 0 & 0 \\ \frac{-i\sqrt{2}\ n_2\cdot q}{q^2\,(q^2+4)} & \frac{(q^2)^2+4(n_1\cdot q)^2}{(q^2)^2(q^2+4)} & \frac{-4n_1\cdot q\,n_2\cdot q}{(q^2)^2(q^2+4)} & 0 & 0 & 0 \\ \frac{-i\sqrt{2}\ n_1\cdot q}{q^2\,(q^2+4)} & \frac{-4n_1\cdot q\,n_2\cdot q}{(q^2)^2(q^2+4)} & \frac{(q^2)^2+4(n_2\cdot q)^2}{(q^2)^2(q^2+4)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2(q^2+2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2(q^2+2)} & \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\,q^2}\,\mathbb{1}_5 \end{pmatrix} \quad \text{(A.3)}$$

The fermionic propagator corresponding to (2.22) is

$$K_F^{-1}=\frac{\Gamma_L}{q^2+1}\Big[in_1\times q(\Gamma^0-\sqrt{2}\Gamma^9)-in_2\times q\Gamma^1+n_1\times n_2\Gamma^{019}\Big]\mathcal{C}^{-1}\,, \quad \text{(A.4)}$$

where $n\times q=\epsilon^{ab}n_a q_b$, $\Gamma_L=\frac{1}{2}(1+\Gamma_{11})$ is the left-handed chiral projector and $\mathcal{C}=\Gamma^0$ is the charge conjugation matrix (for notation see also [7]).

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